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Evidently revolving $\frac{\partial W}{\partial x}$ through 90° gives coincidence with $\frac{\partial W}{\partial y}$, or

$$i\frac{\partial W}{\partial x}=\frac{\partial W}{\partial y},$$

which is Riemann's condition of functionality.

INTEGRATION OF ELLIPTIC INTEGRALS.

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[Continued from October Number.]

Let $e_1=\frac{e^2}{2\pm e^2}$, $e_2=\frac{e_1^2}{2\pm e_1^2}$, $e_3=\frac{e_2^2}{2\pm e_2^2}$, ... , $e_n=\frac{e^2{}_{n-1}}{2\pm e^2{}_{n-1}}$.

$$[(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]^{\frac{1}{2}}=R,$$

$$[(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]^{\frac{1}{2}}=R_1,$$

$$[(1-e_2^2\sin^2 4\theta)(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]^{\frac{1}{2}}=R_2,$$

$$[(1-e_3^2\sin^2 8\theta)(1-e_2^2\sin^2 4\theta)(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]=R_3$$

. . . . . .

$$[(1-e_n^2\sin^2 2^n\theta)(1-e^2{}_{n-1}\sin^2 2^{n-1}\theta)\ldots.(1-e_3^2\sin^2 8\theta)$$

$$[(1-e_2^2\sin^2 4\theta)(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]\ \ldots R_n.$$

But $R=[1\pm e^2(\sin^2\theta+\cos^2\theta)+e^4\sin^2\theta\cos^2\theta]^{\frac{1}{2}}=[1\pm e^2+\frac{1}{4}e^4\sin^2 2\theta]^{\frac{1}{2}}$

$$=\tfrac{1}{2}[(2\pm e^2)^2-e^4\cos^2 2\theta]^{\frac{1}{2}}=\tfrac{1}{2}(2\pm e^2)[1-e_1^2\sin^2(\tfrac{1}{2}\pi-2\theta)]^{\frac{1}{2}}.$$

Let $\frac{1}{2}\pi-2\theta=\varphi$. Then $d\theta=-\frac{1}{2}d\varphi$.

$$\therefore\int_0^{\frac{1}{2}\pi}Rd\theta=\tfrac{1}{4}(2\pm e^2)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sqrt{(1-e_1^2\sin^2\varphi)}d\varphi$$

$$=\tfrac{1}{2}(2\pm e^2)\int_0^{\frac{1}{2}\pi}\sqrt{(1-e_1^2\sin^2\varphi)}d\varphi=\tfrac{1}{2}(2\pm e^2)E(e_1,\ \tfrac{1}{2}\pi)\ldots\ldots\ldots\ldots\ldots\ldots(85).$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{R}=\frac{2}{2\pm e^2}\int_0^{\frac{1}{2}\pi}\frac{d\varphi}{\sqrt{(1-e_1^2\sin^2\varphi)}}=\frac{2}{2\pm e^2}F(e_1,\ \tfrac{1}{2}\pi)\ldots\ldots\ldots\ldots..(86).$$

$$R_1=\tfrac{1}{2}(2\pm e^2)[(1-e_1^2\sin^2 2\theta)(1-e_1^2\cos^2 2\theta)]^{\frac{1}{2}}$$

$$=\tfrac{1}{4}(2\pm e^2)(2-e_1^2)[1-e_2^2\sin^2(\tfrac{1}{2}\pi-4\theta)]^{\frac{1}{2}}\,.$$

Let $(\frac{1}{2}\pi-4\theta)=\lambda$. $\therefore\ d\theta=-\frac{1}{4}d\lambda$.

$$\therefore \int_0^{\frac{1}{2}\pi}R_1\,d\theta=\tfrac{1}{4}\int_{-\frac{1}{2}3\pi}^{\frac{1}{2}\pi}R_1\,d\lambda=\tfrac{1}{4}\int_0^{\frac{1}{2}\pi}R_1\,d\lambda+\tfrac{1}{4}\int_0^{\frac{1}{2}3\pi}R_1\,d\lambda=\tfrac{1}{4}\int_0^{\frac{1}{2}\pi}R_1\,d\lambda$$

$$+\tfrac{3}{4}\int_0^{\frac{1}{2}\pi}R_1\,d\lambda=\int_0^{\frac{1}{2}\pi}R_1\,d\lambda=\tfrac{1}{4}(2\pm e^2)(2-e_1^2)\int_0^{\frac{1}{2}\pi}\sqrt{(1-e_2^2\sin^2\lambda)}\,d\lambda$$

$$=\tfrac{1}{4}(2\pm e^2)(2-e_1^2)E(e_2,\ \tfrac{1}{2}\pi)\ \dots\dots\dots\dots(87).$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\,\theta}{R_1}=4\int_0^{\frac{1}{2}\pi}\frac{d\,\lambda}{R_1}=\frac{4}{(2\pm e^2)(2-e_1^2)}F(e_2,\ \tfrac{1}{2}\pi)\ \dots\dots\dots\dots(88).$$

Similarly, $\int_0^{\frac{1}{2}\pi}R_2\,d\theta=\frac{1}{8}(2\pm e^2)(2-e_1^2)(2-e_2^2)E(e_3,\ \frac{1}{2}\pi)\ \dots\dots\dots\dots(89).$

$$\int_0^{\frac{1}{2}\pi}\frac{d\,\theta}{R_2}=\frac{8}{(2\pm e^2)(2-e_1^2)(2-e_2^2)}F(e_3,\ \tfrac{1}{2}\pi)\ \dots\dots\dots\dots(90).$$

$$\int_0^{\frac{1}{2}\pi}R_3\,d\theta=\frac{1}{2^4}(2\pm e^2)(2-e_1^2)(2-e_2^2)(2-e_3^2)E(e_4,\ \tfrac{1}{2}\pi)\dots\dots\dots\dots(91).$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\,\theta}{R_3}=\frac{2^4}{(2\pm e^2)(2-e_1^2)(2-e_2^2)(2-e_3^2)}F(e_4,\ \tfrac{1}{2}\pi)\dots\dots\dots\dots(92).$$

. . . .

$$\int_0^{\frac{1}{2}\pi}R_n d\theta=\frac{1}{2^{n+1}}(2\pm e^2)(2-e_1^2)(2-e_2^2)\dots.(2-e_n^2)E(e_{n+1},\ \tfrac{1}{2}\pi)\dots\dots(93).$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{R_n}=\frac{2^{n+1}}{(2\pm e^2)(2-e_1^2)(2-e_2^2)\dots.(2-e_n^2)}F(e_{n+1},\ \tfrac{1}{2}\pi)\dots\dots\dots\dots(94).$$

$$\int_0^{\frac{1}{2}\pi}(1-e^2\cos^2\theta)^{\frac{1}{2}(2m+1)}d\theta=\int_0^{\frac{1}{2}\pi}[1-e^2\sin^2(\tfrac{1}{2}\pi-\theta)]^{\frac{1}{2}[2m+1)}d\theta.$$

Let $\frac{1}{2}\pi-\theta=\beta$.

$$\therefore \int_0^{\frac{1}{2}\pi}[1-e^2\sin^2(\tfrac{1}{2}\pi-\theta)]^{\frac{1}{2}(2m+1)}d\theta=\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\beta)^{\frac{1}{2}(2m+1)}d\beta$$

$$=\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\beta)^m\sqrt{(1-e^2\sin^2\beta)}\,d\beta=M,\ \text{suppose.}$$

Let $m=0$. $\therefore\ M=\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\beta)}\,d\beta=E(e,\ \frac{1}{2}\pi)$.

Let $m=1$. $\therefore M=\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\beta)\surd(1-e^2\sin^2\beta)d\beta$.

From (2) and (8), $M=\frac{1}{3}[2(2-e^2)E(e,\ \frac{1}{2}\pi)-(1-e^2)F(e,\ \frac{1}{2}\pi)]$........(95).

Let $m=2$. $\therefore M=\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\beta)^2\surd(1-e^2\sin^2\beta)d\beta$.

From (2), (8), and (15),

$M=\frac{1}{15}[(23-23e^2-8e^4)E(e,\ \frac{1}{2}\pi)-4(2-3e^2+e^4)F(e,\ \frac{1}{2}\pi)]$ ....(96),

and so on for other values of m.

To integrate $\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-e^2\cos^2\theta)^{\frac{1}{2}(2m+1)}}$, let $\tan\delta=\frac{1}{\surd(1-e^2)}\tan\theta$.

$$\therefore \int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-e^2\cos^2\theta)^{\frac{1}{2}(2m+1)}}=\frac{1}{(1-e^2)^m}\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\delta)^{\frac{1}{2}(2m-1)}d\delta,$$

a form like preceding. When $m=0$ we get

$$\int_0^{\frac{1}{2}\pi}\surd(1-e^2\cos^2\theta)d\theta=\int_0^{\frac{1}{2}\pi}\surd(1-e^2\sin^2\theta)d\theta=E(e,\ \tfrac{1}{2}\pi).$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\surd(1-e^2\cos^2\theta)}=\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\surd(1-e^2\sin^2\theta)}=F(e,\ \tfrac{1}{2}\pi).$$

By letting $\cot\gamma=\frac{1}{\surd(1-e^2)}\cot\theta$, we get

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2m+1)}}=\frac{1}{(1-e^2)^m}\int_0^{\frac{1}{2}\pi}(1-e^2\cos^2\gamma)^{\frac{1}{2}(2m-1)}d\gamma,\text{ a previous form.}$$

Letting $\tan^2\delta=\frac{1}{1-e^2}\tan^2\theta$, we get

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-c\cos^2\theta)(1-e^2\cos^2\theta)^{\frac{1}{2}(2m+1)}}.$$

$$=\frac{1}{(1-c)(1-e^2)^m}\int_0^{\frac{1}{2}\pi}\frac{(1-e^2\sin^2\delta)^{m+1}d\delta}{(1+\frac{c-e^2}{1-c}\sin^2\delta)\surd(1-e^2\sin^2\delta)}.$$

We will now proceed to more general work.

$$\int\frac{\surd(1-e^2\sin^2\theta)d\theta}{\sin^2\theta}=-\cot\theta\surd(1-e^2\sin^2\theta)-e^2\int\frac{\cos^2\theta d\theta}{\surd(1-e^2\sin^2\theta)}$$

$$=(1-e^2)F(e,\ \theta)-E(e,\ \theta)-\cot\theta\surd(1-e^2\sin^2\theta)=u_1\ \ldots.(97).$$

$$\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^2\theta}=\tan\theta\sqrt{(1-e^2\sin^2\theta)}+e^2\int\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=F(e,\ \theta)-E(e,\ \theta)+\tan\theta\sqrt{(1-e^2\sin^2\theta)}=R_1\ \ldots\ldots\ldots\ldots\ldots\ldots(98).$$

$$\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}=e^2\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^2\theta}$$

$$=F(e,\ \theta)-E(e,\ \theta)-\cot\theta\sqrt{(1-e^2\sin^2\theta)}=V_1\ \ldots\ldots\ldots\ldots\ldots\ldots(99).$$

$$\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{1-e^2}\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^2\theta}-\frac{e^2}{1-e^2}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=F(e,\ \theta)-\frac{1}{1-e^2}E(e,\ \theta+\frac{\tan\theta}{1-e^2}\sqrt{(1-e^2\sin^2\theta)}=S_1\ \ \ldots\ \ldots.(100).$$

$$\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^4\theta}=-\frac{\cos\theta}{3}\left(\frac{1}{\sin^3\theta}+\frac{2}{\sin\theta}\right)\sqrt{(1-e^2\sin^2\theta)}+\frac{e^2}{3}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{e^2}{3}\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}-\frac{2e^2}{3}\int\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{2(1-e^2)}{3}F(e,\ \theta)-$$

$$\frac{(2-e^2)}{3}E(e,\ \theta)-\frac{(2-e^2)}{3}\cot\theta\sqrt{(1-e^2\sin^2\theta)}-\frac{\cos\theta}{3\sin^3\theta}\sqrt{(1-e^2\sin^2\theta)}=U_2\ ..(101).$$

$$\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^4\theta}=\frac{\sin\theta}{3}\left(\frac{1}{\cos^3\theta}+\frac{2}{\cos\theta}\right)\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{e^2}{3}\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}-\frac{e^3}{3}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+\frac{2e^2}{3}\int\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\tfrac{2}{3}F(e,\ \theta)-\frac{(2-e^2)}{3(1-e^2)}E(e,\ \theta)+\frac{(2-e^2)}{3(1-e^2)}\tan\theta\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{\sin\theta}{3\cos^3\theta}\sqrt{(1-e^3\sin^2\theta)}=R_2\ \ \ldots\ldots\ldots\ldots\ldots(102).$$

$$\int\frac{d\theta}{\sin^4\theta\sqrt{(1-e^2\sin^2\theta)}}=e^2\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}+\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^4\theta}$$

$$=\frac{(2+e^2)}{3}F(e,\ \theta)-\frac{2(1+e^2)}{3}E(e,\ \theta)-\frac{2(1+e^2)}{3}\cot\theta\sqrt{(1-e^2\sin^2\theta)}$$

$$-\frac{\cos\theta}{3\sin^3\theta}\sqrt{(1-e^2\sin^2\theta)}=V_2\ \ \ldots\ldots\ldots\ldots\ldots(103).$$

$$\int\frac{d\theta}{\cos^4\theta\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{1-e^2}\int\frac{\sqrt{(1-e^2\sin^2\theta)}\,d\theta}{\cos^4\theta}-\frac{e^2}{1-e^2}\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{(2-3e^2)}{3(1-e^2)}F(e,\ \theta)-\frac{2(1-2e^2)}{3(1-e^2)^2}E(e,\ \theta)+\frac{2(1-2e^2)}{3(1-e^2)^2}\tan\theta\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{\sin\theta}{3(1-e^2)\cos^3\theta}\sqrt{(1-e^2\sin^2\theta)}=S_2\ \ldots\ldots\ldots\ldots(104).$$

$$\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^6\theta}=-\frac{\cos\theta}{5}\left(\frac{1}{\sin^5\theta}+\frac{4}{3\sin^3\theta}+\frac{8}{3\sin\theta}\right)\sqrt{(1-e^2\sin^2\theta)}$$

$$-\frac{e^2}{5}\int\frac{d\theta}{\sin^4\theta\sqrt{(1-e^2\sin^2\theta)}}-\frac{e^2}{15}\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}+\frac{4e^2}{15}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{8e^2}{15}\int\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{(8-7e^2-e^4)}{15}F(e,\ \theta)-\frac{(8-3e^2-2e^4)}{15}E(e,\ \theta)$$

$$-\frac{(8-3e^2-2e^4)}{15}\cot\theta\sqrt{(1-e^2\sin^2\theta)}-\frac{(4-e^2)}{15}\frac{\cos\theta}{\sin^3\theta}\sqrt{(1-e^2\sin^2\theta)}$$

$$-\frac{\cos\theta}{5\sin^5\theta}\sqrt{(1-e^2\sin^2\theta)}=U_3\ \ldots\ldots\ldots\ldots\ldots\ldots(105).$$

$$\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^6\theta}=\frac{\sin\theta}{5}\left(\frac{1}{\cos^5\theta}+\frac{4}{3\cos^3\theta}+\frac{8}{3\cos\theta}\right)\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{e^2}{5}\int\frac{d\theta}{\cos^4\theta\sqrt{(1-e^2\sin^2\theta)}}+\frac{e^2}{15}\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}-\frac{4e^2}{15}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$+\frac{8e^2}{15}\int\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{(8-9e^2)}{15(1-e^2)}F(e,\ \theta)-\frac{(8-13e^2+3e^4)}{15(1-e^2)^2}E(e,\ \theta)$$

$$+\frac{8-13e^2+3e^4}{15(1-e^2)^2}\tan\theta\sqrt{(1-e^2\sin^2\theta)}+\frac{4-3e^2}{15(1-e^2)}\cdot\frac{\sin\theta}{\cos^3\theta}\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{\sin\theta}{5\cos^5\theta}\sqrt{(1-e^2\sin^2\theta)}=R_3\ \ldots\ldots\ldots\ldots\ldots\ldots(106).$$

$$\int\frac{d\theta}{\sin^6\theta\sqrt{(1-e^2\sin^2\theta)}}=e^2\int\frac{d\theta}{\sin^4\theta\sqrt{(1-e^2\sin^2\theta)}}+\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^6\theta}$$

$$=\frac{(8+3e^2+4e^4)}{15}F(e,\ \theta)-\frac{(8+7e^2+8e^4)}{15}E(e,\ \theta)-\frac{\cos\theta}{5\sin^5\theta}\sqrt{(1-e^2\sin^2\theta)}$$

$$-\frac{(8+7e^2+8e^4)}{15}\tan\theta\sqrt{(1-e^2\sin^2\theta)}-\frac{4(1+e^2)}{15}\cdot\frac{\cos\theta}{\sin^3\theta}\sqrt{(1-e^2\sin^2\theta)}=V_3\ \ldots(107).$$

[To be Concluded.]